Why do Random Matrices Explain Learning? An Argument of Universality Offered by GANs

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Abstract

Context:

- Study of large **Gram** matrices for concentrated data.

Motivation:

- **Observation:** RMT predicts ML performances in high-dimension under Gaussian assumptions on data.
- **BUT** Real data **are unlikely close** to Gaussian vectors.
- Gaussian vectors fall within a **larger**, more **useful**, class of random vectors.

Results:

- GAN data [1] fall within the class of concentrated vectors.
- Only first and second order statistics of concentrated data matter to describe the behavior of **Gram** matrices.

Concentrated Vectors

Definition 1. Given a normed space $(E,\|\cdot\|_E)$ and $q\in\mathbb{R}$, a random vector $X\in E$ is *q-exponentially concentrated* if for any 1-**Lipschitz function** $\mathcal{F}:\mathbb{R}^p\to\mathbb{R}$, there exists $C,c>0$ s.t.

$$\forall t>0,\ \mathbb{P}\left\{|\mathcal{F}(X)-\mathbb{E}\mathcal{F}(X)|\geq t\right\}\leq C\,e^{-ct^q}\ \xrightarrow{\text{denoted}}\ \boxed{X\in\mathcal{O}(e^{-\cdot^q})\text{ in }(E,\|\cdot\|_E)}$$

(P1) $X\sim\mathcal{N}(0,I_p)$ is 2-exponentially concentrated [2].

(P2) If $X\in\mathcal{O}(e^{-\cdot^q})$ and $\mathcal{G}$ is ℓ-**Lipschitz**, then $\mathcal{G}(X)\in\mathcal{O}(e^{-(\cdot/\ell)^q})$.

"Concentrated vectors are stable through Lipschitz maps."

Model & Assumptions

(A1) **Data matrix** (distributed in k classes $\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_k$):

$$X=\left[\underbrace{x_1,\ldots,x_{n_1}}_{\in\mathcal{O}(e^{-\cdot^{q_1}})},\underbrace{x_{n_1+1},\ldots,x_{n_2}}_{\in\mathcal{O}(e^{-\cdot^{q_2}})},\ldots,\underbrace{x_{n-n_k+1},\ldots,x_n}_{\in\mathcal{O}(e^{-\cdot^{q_k}})}\right]\in\mathbb{R}^{p\times n}$$

Model statistics:

$$\mu_\ell=\mathbb{E}_{x_i\in\mathcal{C}_\ell}[x_i],\quad C_\ell=\mathbb{E}_{x_i\in\mathcal{C}_\ell}[x_ix_i^\intercal]$$

(A2) **Growth rate assumptions:** As $p\to\infty$,

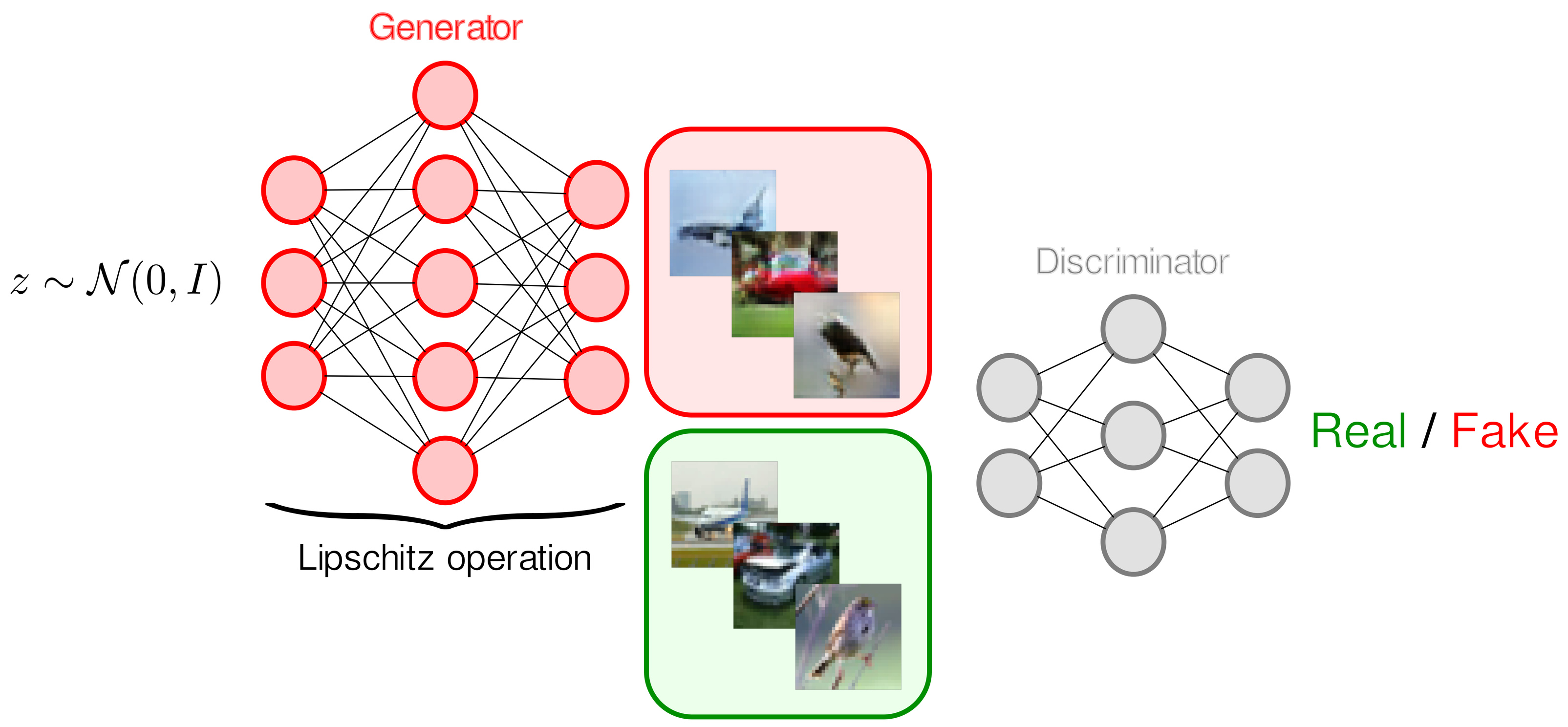
1. $p/n\to c\in(0,\infty)$.
2. The number of classers k is bounded.
3. For any $\ell\in[k]$, $\|\mu_\ell\|=\mathcal{O}(\sqrt{p})$.

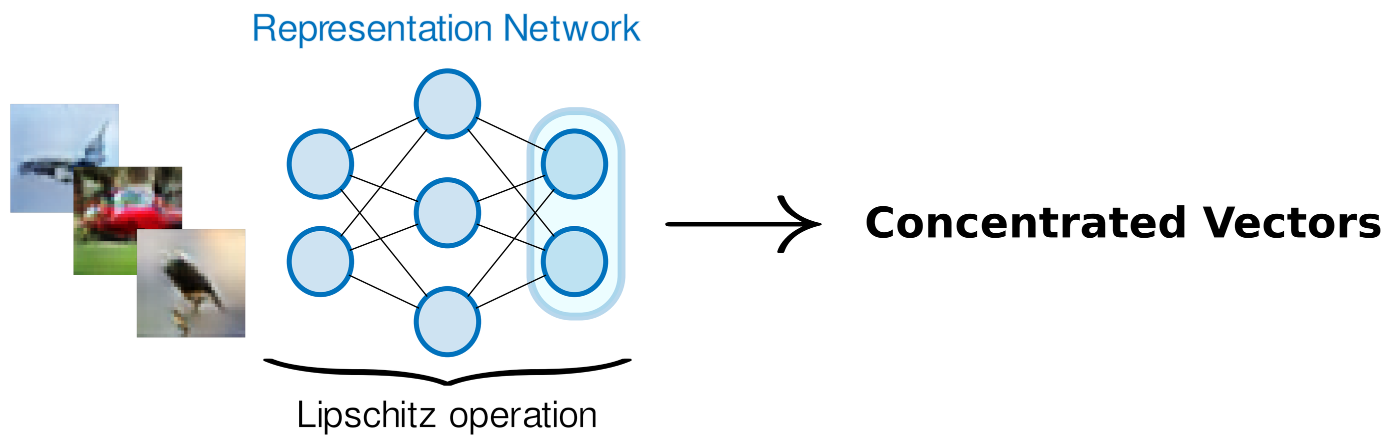
Gram matrix and its resolvent:

$$\boxed{G=\frac{1}{p}X^\intercal X,\ Q(z)=(G+zI_n)^{-1}}$$

$$m_L(z)=\frac{1}{n}\operatorname{tr}(Q(-z)),\ UU^\intercal=\frac{1}{2\pi i}\oint_\gamma Q(-z)dz$$

Why Concentrated Vectors?





Deterministic Equivalent for $Q(z)$

Theorem Under the assumptions (A1) and (A2).
We have $Q(z)\in\mathcal{O}(e^{-(\sqrt{p}\cdot)^q})$ in $(\mathbb{R}^{n\times n},\|\cdot\|)$. Furthermore,

$$\left\|\mathbb{E}[Q(z)]-\tilde{Q}(z)\right\|=\mathcal{O}\left(\sqrt{\frac{\log p}{p}}\right)\quad\text{where }\tilde{Q}(z)=\frac{1}{z}\Lambda(z)+\frac{1}{p\,z}J\Omega(z)J^\intercal$$

with $\Lambda(z)=\left\{\frac{1_{n_\ell}}{1+\delta_\ell(z)}\right\}_{\ell=1}^k$ and $\Omega(z)=\{\mu_\ell^\intercal\tilde{R}(z)\mu_\ell\}_{\ell=1}^k$

$$\tilde{R}(z)=\left(\frac{1}{k}\sum_{\ell=1}^k\frac{C_\ell}{1+\delta_\ell(z)}+zI_p\right)^{-1}$$

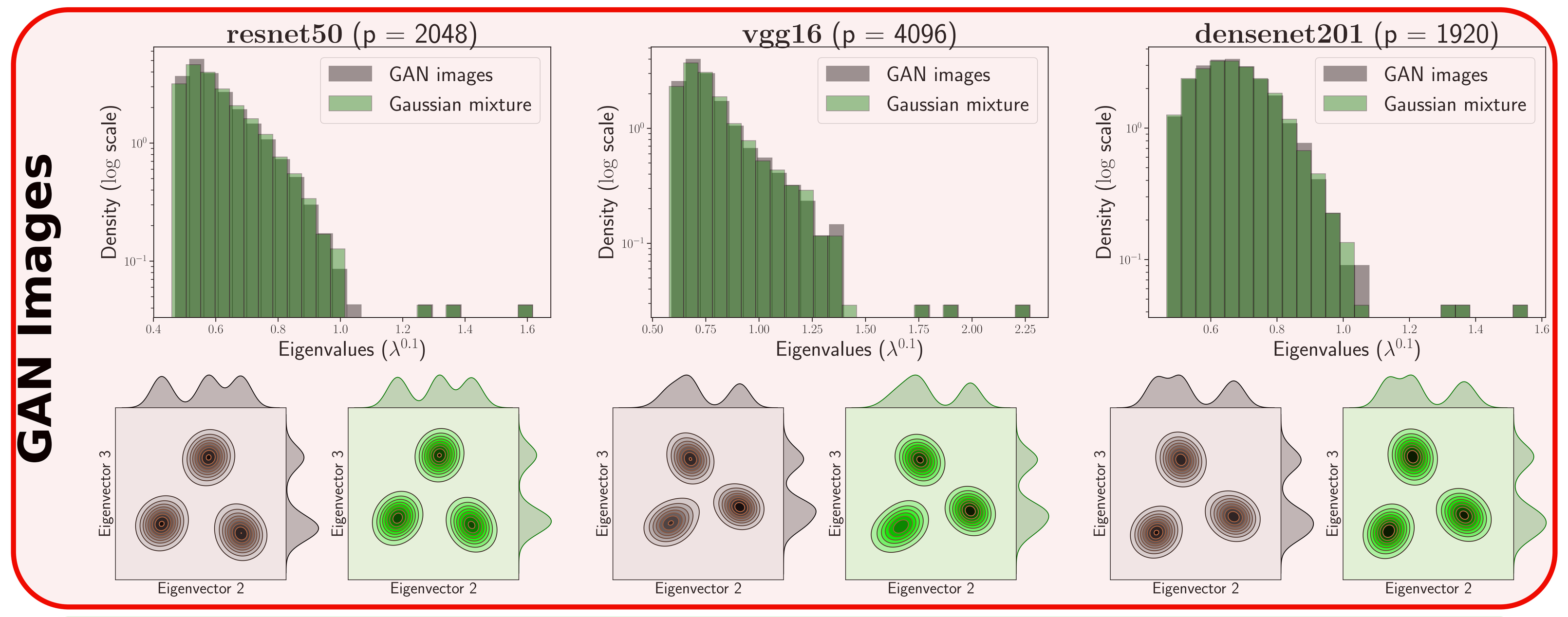
with $\delta(z)=[\delta_1(z),\ldots,\delta_k(z)]$ is the unique fixed point of the system of equations

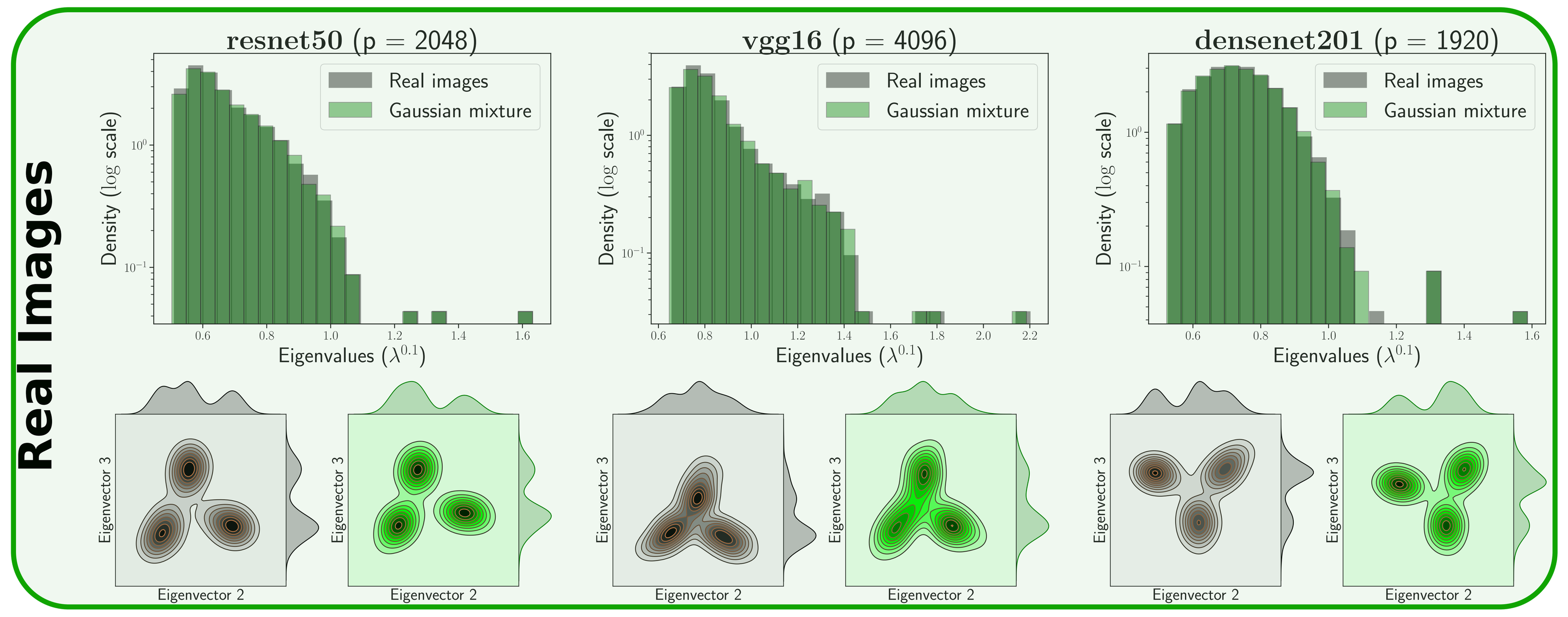
$$\delta_\ell(z)=\frac{1}{p}\operatorname{tr}\left(C_\ell\left(\frac{1}{k}\sum_{j=1}^k\frac{C_j}{1+\delta_j(z)}+zI_p\right)^{-1}\right)\quad\text{for each }\ell\in[k].$$

Key Observation: Only first and second order statistics matter!

Application to GAN-Generated Images

Figure 1. Images generated by the BigGAN model [3].





Perspectives

- **Generalize** to other ML tasks (Classification, Regression and TL).
- **Understand** and **improve** GANs by adding statistic constraints.

References

[1] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio, "Generative adversarial nets", in NIPS 2014.

[2] Terence Tao, "Topics in random matrix theory, volume 132". American Mathematical Society Providence, RI, 2012.

[3] Andrew Brock, Jeff Donahue, and Karen Simonyan, "Large scale GAN training for high fidelity image synthesis", in ICLR 2019.

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