On the Accuracy of Hotelling-Type Asymmetric Tensor Deflation: A Random Tensor Analysis

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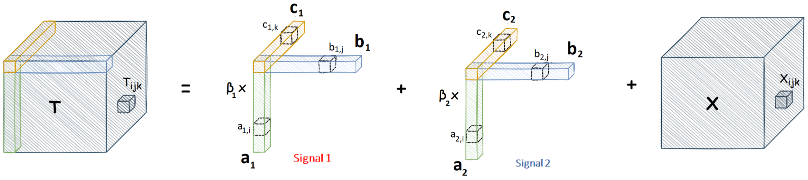
Introduction: Asymmetric Rank-2 Spiked Tensor Model



Consider the rank-2 spiked tensor model:

$$\mathbf{T} = \underbrace{\beta_1 \mathbf{a}_1 \otimes \mathbf{b}_1 \otimes \mathbf{c}_1}_{\text{signal 1}} + \underbrace{\beta_2 \mathbf{a}_2 \otimes \mathbf{b}_2 \otimes \mathbf{c}_2}_{\text{signal 2}} + \frac{1}{\sqrt{n_1 + n_2 + n_3}} \underbrace{\mathbf{X}}_{\text{noise}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$$

where $\beta_t \geq 0$, $\|\mathbf{a}_t\| = \|\mathbf{b}_t\| = \|\mathbf{c}_t\| = 1$ for $t = 1, 2$, $X_{ijk} \sim \mathcal{N}(0, 1)$ i.i.d.

- Assume **correlated** signals:

$$\alpha_a := \langle \mathbf{a}_1, \mathbf{a}_2 \rangle \neq 0, \quad \alpha_b := \langle \mathbf{b}_1, \mathbf{b}_2 \rangle \neq 0, \quad \alpha_c := \langle \mathbf{c}_1, \mathbf{c}_2 \rangle \neq 0$$

- We focus on **deflation algorithms** that compute successive rank-1 approximations

Introduction: Hotelling-type Tensor Deflation

Successive best rank-1 approximations (here for rank $r = 2$).

- Input: Observed tensor $\mathbf{T}$, Output: Rank-1 tensors $\boldsymbol{u}_t \otimes \boldsymbol{v}_t \otimes \boldsymbol{w}_t$, $t = 1, 2, \ldots$
- Step $t = 1$
  - Best rank-1 tensor approximation of $\mathbf{T}_1 = \mathbf{T}$:

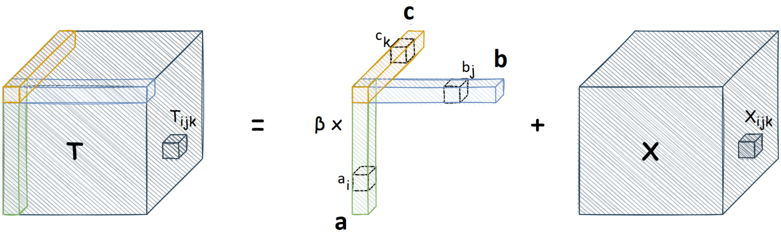
$$(\hat{\lambda}_1, \hat{\boldsymbol{u}}_1, \hat{\boldsymbol{v}}_1, \hat{\boldsymbol{w}}_1) = \underset{\lambda_1 > 0, \|\boldsymbol{u}_1\| = \|\boldsymbol{v}_1\| = \|\boldsymbol{w}_1\| = 1}{\arg\min} \|\mathbf{T}_1 - \lambda_1 \boldsymbol{u}_1 \otimes \boldsymbol{v}_1 \otimes \boldsymbol{w}_1\|_F^2$$

- Step $t = 2$
  - Best rank-1 tensor approximation of $\mathbf{T}_2 = \mathbf{T} - \hat{\lambda}_1 \hat{\boldsymbol{u}}_1 \otimes \hat{\boldsymbol{v}}_1 \otimes \hat{\boldsymbol{w}}_1$:

$$(\hat{\lambda}_2, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{v}}_2, \hat{\boldsymbol{w}}_2) = \underset{\lambda_2 > 0, \|\boldsymbol{u}_2\| = \|\boldsymbol{v}_2\| = \|\boldsymbol{w}_2\| = 1}{\arg\min} \|\mathbf{T}_2 - \lambda_2 \boldsymbol{u}_2 \otimes \boldsymbol{v}_2 \otimes \boldsymbol{w}_2\|_F^2$$

- Deflation yields wrong results for non-orthogonal tensors, even in the absence of noise (the Eckart-Young theorem does not apply here)
- This work is about **quantifying the error** in terms of alignment between true and estimated singular vectors: $\langle \mathbf{a}_t, \boldsymbol{u}_t \rangle, \langle \mathbf{b}_t, \boldsymbol{v}_t \rangle, \langle \mathbf{c}_t, \boldsymbol{w}_t \rangle$
- As a side product, we obtain a powerful estimator for the β's

Asymmetric Rank-1 Spiked Tensor Model



Let us take a step backward and consider a spiked rank-1 model

$$\mathbf{T} = \underbrace{\beta \mathbf{a} \otimes \mathbf{b} \otimes \boldsymbol{c}}_{\text{signal}} + \frac{1}{\sqrt{n_1 + n_2 + n_3}} \underbrace{\mathbf{X}}_{\text{noise}} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$$

where $\beta \geq 0$, $\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\| = 1$, $X_{ijk} \sim \mathcal{N}(0, 1)$ i.i.d.

Random Matrix Approach (Seddik et al., 2023)

- A traditional estimator of the rank-1 signal is the Maximum Likelihood Estimator (MLE)

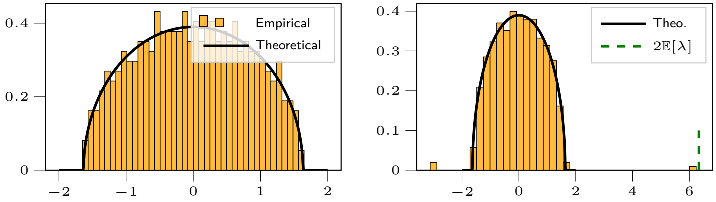
$$(\hat{\lambda}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}) = \underset{\lambda>0, \|\boldsymbol{u}\|=\|\boldsymbol{v}\|=\|\boldsymbol{w}\|=1}{\arg\min} \|\mathbf{T} - \lambda \boldsymbol{u} \otimes \boldsymbol{v} \otimes \boldsymbol{w}\|_F^2$$

The critical points of the likelihood are tensor singular vectors and satisfy **(Lim, 2005)**:

$$\underbrace{\begin{pmatrix} \mathbf{0}_{\mathbf{n_1}\times\mathbf{n_1}} & \mathbf{T}(\cdot,\cdot,\boldsymbol{w}) & \mathbf{T}(\cdot,\boldsymbol{v},\cdot) \\ \mathbf{T}(\cdot,\cdot,\boldsymbol{w})^T & \mathbf{0}_{\mathbf{n_2}\times\mathbf{n_2}} & \mathbf{T}(\boldsymbol{u},\cdot,\cdot) \\ \mathbf{T}(\cdot,\boldsymbol{v},\cdot)^T & \mathbf{T}(\boldsymbol{u},\cdot,\cdot)^T & \mathbf{0}_{\mathbf{n_3}\times\mathbf{n_3}} \end{pmatrix}}_{\mathbf{\Phi}_3(\mathbf{T},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = 2\lambda \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix}$$

where we have the contractions $(\mathbf{T}(\cdot,\cdot,\boldsymbol{w}))_{ij} = \sum_{k=1}^{n_3} w_k T_{ijk}$.

- We establish the asymptotic (large dimensional) properties of these critical points by studying the spectrum of $\mathbf{\Phi}_3(\mathbf{T}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ using Random Matrix Theory
- Spectrum of $\Phi_3(\mathbf{T}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ for random $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ (left) and for tensor singular vectors (right):



Asymptotic Spectral Norm and Alignments

Assumption 1. As $n_i \to \infty$ with $\frac{n_i}{\sum_j n_j} \to c_i \in (0,1)$, there exists a sequence of critical points $(\hat{\lambda}^{(n)}, \hat{\boldsymbol{u}}^{(n)}, \hat{\boldsymbol{v}}^{(n)}, \hat{\boldsymbol{w}}^{(n)})$ s.t.

$$\begin{cases} \hat{\lambda}^{(n)} \xrightarrow{\text{a.s.}} \lambda > \lambda_{\mathsf{min}} \\ |\langle \mathbf{a}^{(n)}, \hat{\boldsymbol{u}}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_a > 0 \\ |\langle \mathbf{b}^{(n)}, \hat{\boldsymbol{v}}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_b > 0 \\ |\langle \mathbf{c}^{(n)}, \hat{\boldsymbol{w}}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_c > 0 \end{cases}$$

Theorem 1 (SGC'21). Under Assumption 1, there exists $\beta_s > 0$ such that for all $\beta > \beta_s$ the following fixed point equation holds

$$\rho_a = q_a(\lambda), \quad \rho_b = q_b(\lambda), \quad \rho_c = q_c(\lambda), \qquad \lambda = \beta \rho_a \rho_b \rho_c - g(\lambda)$$

where $g(\lambda) = g_a(\lambda) + g_b(\lambda) + g_c(\lambda)$ and

$$\begin{cases} g_a^2(z) - (g(z) + z) g_a(z) - c_1 = 0 \quad , \quad q_a(z) = \sqrt{1 - g_a^2(z)/c_1} \\ g_b^2(z) - (g(z) + z) g_b(z) - c_2 = 0 \quad , \quad q_b(z) = \sqrt{1 - g_b^2(z)/c_2} \\ g_c^2(z) - (g(z) + z) g_c(z) - c_3 = 0 \quad , \quad q_c(z) = \sqrt{1 - g_c^2(z)/c_3} \end{cases}$$

Back to Hotelling Deflation: Spectral Measure

For both deflation steps:

$$\mathsf{T}_t \quad \to \quad \text{analyse spectrum of } \boldsymbol{\Phi}_3(\mathsf{T}_t, \hat{\boldsymbol{u}}_t, \hat{\boldsymbol{v}}_t, \hat{\boldsymbol{w}}_t)$$

Assumption 2. As $n_i \to \infty$ with $\frac{n_i}{\sum_j n_j} \to c_i \in (0, 1)$, there exists a sequence of critical points $(\hat{\lambda}_t^{(n)}, \hat{\boldsymbol{u}}_t^{(n)}, \hat{\boldsymbol{v}}_t^{(n)}, \hat{\boldsymbol{w}}_t^{(n)})$ s.t.

$$\begin{cases} \hat{\lambda}_t^{(n)} \xrightarrow{\text{a.s.}} \lambda_t > \lambda_{\mathsf{min},t} \\ |\langle \mathbf{a}_t^{(n)}, \hat{\boldsymbol{u}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_{tt',a} > 0, \quad |\langle \hat{\boldsymbol{u}}_t^{(n)}, \hat{\boldsymbol{u}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \eta_{tt',a} > 0 \\ |\langle \mathbf{b}_t^{(n)}, \hat{\boldsymbol{v}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_{tt',b} > 0, \quad |\langle \hat{\boldsymbol{v}}_t^{(n)}, \hat{\boldsymbol{v}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \eta_{tt',b} > 0 \\ |\langle \mathbf{c}_t^{(n)}, \hat{\boldsymbol{w}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \rho_{tt',c} > 0, \quad |\langle \hat{\boldsymbol{w}}_t^{(n)}, \hat{\boldsymbol{w}}_{t'}^{(n)} \rangle| \xrightarrow{\text{a.s.}} \eta_{tt',c} > 0 \end{cases}$$

Hotelling Deflation: Asymptotic Spectral Norm and Alignments

Theorem 2. Under Assumption 2, with $n_1 = n_2 = n_3$ and $\alpha_a = \alpha_b = \alpha_c = \alpha$, then it holds

- $\rho_{tt',a} = \rho_{tt',b} = \rho_{tt',c} = \rho_{tt'}$ for $1 \leq t, t' \leq 2$
- $\eta_{12,a} = \eta_{12,b} = \eta_{12,c} = \eta$

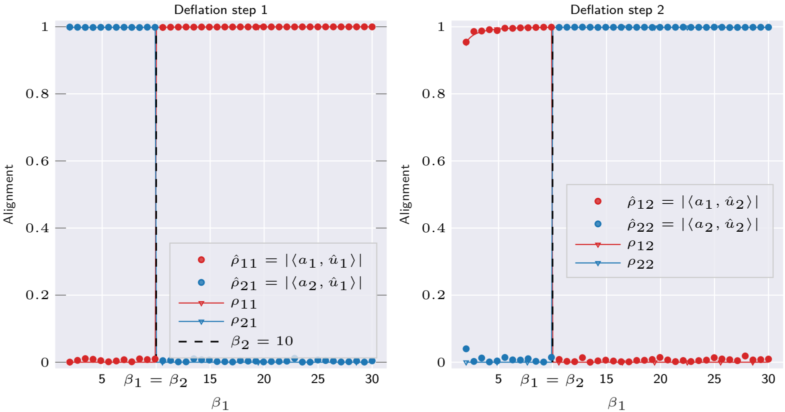
and

$$\begin{cases} \boxed{\begin{array}{ccc} f(\lambda_1) & -\beta_1\rho_{11}^3 & -\beta_2\rho_{21}^3 = 0 \\ h(\lambda_1)\rho_{11} & -\beta_1\rho_{11}^2 & -\beta_2\alpha\rho_{21}^2 = 0 \\ h(\lambda_1)\rho_{21} & -\beta_1\alpha\rho_{11}^2 & -\beta_2\rho_{21}^2 = 0 \end{array}} & \text{Step 1 of deflation} \\ \boxed{\begin{array}{ccc} f(\lambda_2) + \lambda_1\eta^3 & -\beta_1\rho_{12}^3 & -\beta_2\rho_{22}^3 = 0 \\ h(\lambda_2)\rho_{12} + \lambda_1\rho_{11}\eta^2 & -\beta_1\rho_{12}^2 & -\beta_2\alpha\rho_{22}^2 = 0 \\ h(\lambda_2)\rho_{22} + \lambda_1\rho_{21}\eta^2 & -\beta_1\alpha\rho_{12}^2 & -\beta_2\rho_{22}^2 = 0 \\ h(\lambda_2)\eta + q(\lambda_1)\eta^2 & -\beta_1\rho_{11}\rho_{12}^2 & -\beta_2\rho_{21}\rho_{22}^2 = 0 \end{array}} & \text{Step 2 of deflation} \end{cases}$$

where $h(z) = \frac{-1}{g(z)}$ and $q(z) = z + \frac{g(z)}{3}$ and $f(z) = z + g(z)$.

Illustration of Signal Recovery with Deflation: Uncorrelated case

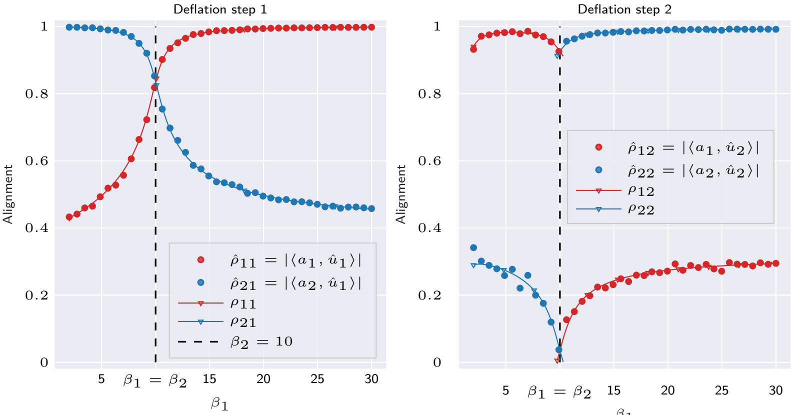
- Dimensions $n_1 = n_2 = n_3 = 50$
- Uncorrelated case: $\boldsymbol{\alpha} = \mathbf{0}$ with a fixed $\beta_2 = 10$



- If $\beta_1 < \beta_2$: signal 2 is recovered at step 1, signal 1 is recovered at step 2
- If $\beta_1 > \beta_2$: signal 1 is recovered at step 1, signal 2 is recovered at step 2

Illustration of Signal Recovery with Deflation: Correlated case

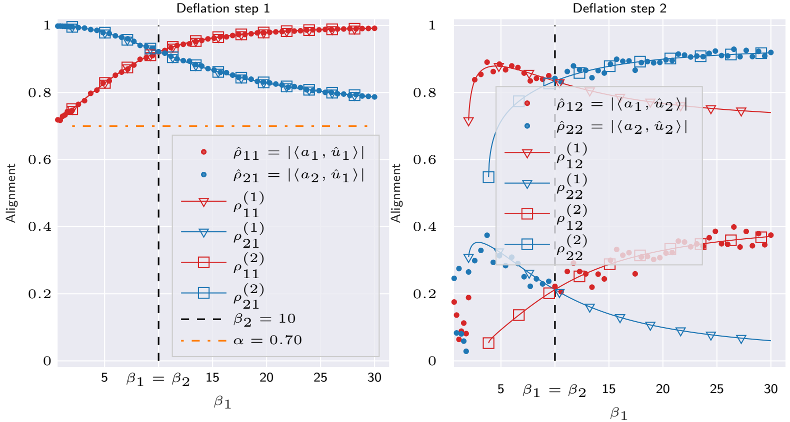
- Dimensions $n_1 = n_2 = n_3 = 50$
- Correlated case: $\boldsymbol{\alpha} = \mathbf{0.4}$ with a fixed $\beta_2 = 10$



- If $\beta_1 \ll \beta_2$: signal 2 is recovered at step 1, signal 1 is recovered at step 2
- If $\beta_1 \approx \beta_2$: **Interference regime**, the output of the deflation is correlated with both signals
- If $\beta_1 \gg \beta_2$: signal 1 is recovered at step 1, signal 2 is recovered at step 2

Illustration of Signal Recovery with Deflation: Correlated case

- Dimensions $n_1 = n_2 = n_3 = 50$
- Correlated case: $\boldsymbol{\alpha} = \mathbf{0.7}$ with a fixed $\beta_2 = 10$



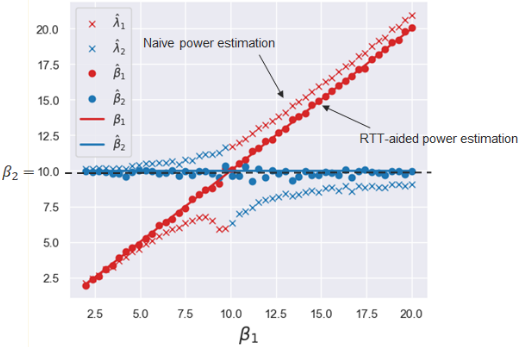
- For $\beta_1 \gtrapprox 4$, the set of fixed point equations admits two solutions ($\cdot^{(1)}$ and $\cdot^{(2)}$) corresponding to multiple critical points of the MLE

RTT-aided Power estimation

- Naive estimator:
  - Use the $\hat{\lambda}_i$ from the deflation as estimates for the true powers β_i (up to reordering)
- Better estimator using random tensor theory:
  - The system of equations from Theorem 1 links the true (β_1, β_2) and (improperly) estimated (λ_1, λ_2) power terms, in the aymptotic regime of large dimensions
  - $\hat{\beta}_1, \hat{\beta}_2$ are obtained by solving the asymptotic system of equations where we plug $\hat{\lambda}_t$ in place of λ_t

$$\underbrace{\boldsymbol{\Psi}}_{\text{7 equations system}}\Big(\underbrace{\hat{\lambda}_1, \hat{\lambda}_2, \hat{\eta},}_{\text{Depend on deflation outputs}}\ \underbrace{\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha}, \hat{\rho}_{11}, \hat{\rho}_{12}, \hat{\rho}_{21}, \hat{\rho}_{22}}_{\text{Solution of the equation system}}\Big) = \mathbf{0}$$

- Numerical comparison: $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$ vs. β_1 for fixed $\beta_2 = 10$



Take Away Messages

- Hotelling-type deflation fails to properly recover low-rank non-orthogonal tensor signals
- RMT was used to characterize the alignments between **true signals** and **deflation outputs** in the asymptotic dimension regime
- This enables a more accurate signal power estimation algorithm

Ongoing work:

- Study the **existence** and **uniqueness** of the solutions of the asymptotic fixed point equations.
- Study RTT-aided signals estimators by "unbiasing" deflation outputs

Thank you for your attention!

Backup slides

Spectral Measure of $\Phi_3(\mathbf{T}, \hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{u}}_3)$

Stieltjes Transform. The Stieltjes transform of a probability measure ν is $g_\nu(z) = \int \frac{d\nu(\lambda)}{\lambda - z}$, $z \in \mathbb{C} \setminus \mathcal{S}(\nu)$.

Definition 1. Let ν by the probability measure with Stieltjes transform $g(z) = g_a(z) + g_b(z) + g_c(z)$ verifying $\Im[g(z)] > 0$ for $\Im[z] > 0$, where $g_a(z)$ satisfies $g_a^2(z) - (g(z) + z) g_a(z) - c_1 = 0$, for $z \notin \mathcal{S}(\nu)$.

Assumption 1. As $n_i \to \infty$ with $\frac{n_i}{\sum_j n_j} \to c_i \in (0, 1)$, there exists a sequence of critical points $(\hat{\lambda}, \hat{\boldsymbol{u}}^{(n)}, \hat{\boldsymbol{v}}^{(n)}, \hat{\boldsymbol{w}}^{(n)})$ s.t. $\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda$, $|\langle \mathbf{a}_i, \hat{\boldsymbol{u}} \rangle| \xrightarrow{\text{a.s.}} \rho_a$ with $\lambda \notin \mathcal{S}(\nu)$ and $\rho_a > 0$.

Theorem 1 (SGC'21). Under Assumption 1, the empirical spectral measure of $\Phi_3(\mathbf{T}, \hat{\boldsymbol{u}}, \boldsymbol{v}, \hat{\boldsymbol{w}})$ converges weakly to ν defined in Definition 1, i.e.

$$\frac{1}{N} \text{tr} \left(\Phi_d(\mathbf{T}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}) - z \mathbf{I}_{N \times N} \right)^{-1} \xrightarrow{\text{a.s.}} g(z)$$

Random Matrix Approach (Goulart et al., 2022)

The optimization problem of maximum likelihood estimator (MLE) for $d = 3$:

$$\min_{\lambda > 0,\ \|\boldsymbol{u}\| = 1} \left\| \mathbf{Y} - \lambda \boldsymbol{u}^{\otimes 3} \right\|_F^2 \quad \Leftrightarrow \quad \max_{\|\boldsymbol{u}\| = 1} \langle \mathbf{Y}, \boldsymbol{u} \otimes \boldsymbol{u} \otimes \boldsymbol{u} \rangle$$

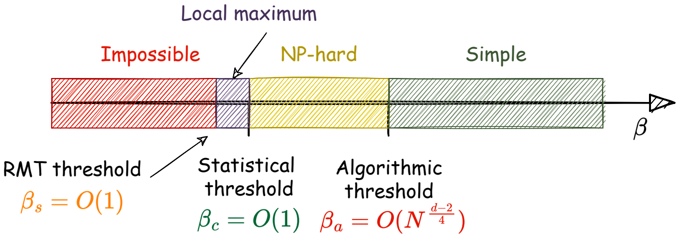
The critical points satisfy **(Lim, 2005)**:

$$\mathbf{Y}(\boldsymbol{u}, \boldsymbol{u}) = \lambda \boldsymbol{u} \quad \Leftrightarrow \quad \mathbf{Y}(\boldsymbol{u})\boldsymbol{u} = \lambda \boldsymbol{u}, \quad \|\boldsymbol{u}\| = 1$$

where $(\mathbf{Y}(\boldsymbol{u}, \boldsymbol{u}))_i = \sum_{jk} u_j u_k Y_{ijk}$ et $(\mathbf{Y}(\boldsymbol{u}))_{ij} = \sum_k u_k Y_{ijk}$. The MLE $\hat{\boldsymbol{x}}$ corresponds to the dominant eigenvector of $\mathbf{Y}(\hat{\boldsymbol{x}})$: $\mathbf{Y}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} = \|\mathbf{Y}\|\hat{\boldsymbol{x}}$.

Hence, the approach from **(Goulart et al., 2021)** consists in studying:

$$\mathbf{Y}(\boldsymbol{u}) = \beta \langle \boldsymbol{x}, \boldsymbol{u} \rangle \boldsymbol{x}\boldsymbol{x}^\top + \frac{1}{\sqrt{N}} \mathbf{W}(\boldsymbol{u}) \in \mathbb{R}^{N \times N}$$



Decomposition Algorithms and Complexity

$$\min_{\lambda>0,\,\|\boldsymbol{u}_i\|=1} \|\mathbf{T} - \lambda \boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_d\|_F^2 \Rightarrow$$ NP-hard **(Hillar et al., 2013)**

- Tensor unfolding: $\mathcal{M}_i(\mathbf{T}) = \beta \boldsymbol{x}_i \boldsymbol{y}_i^\top + \frac{1}{\sqrt{n}} \mathcal{M}_i(\mathbf{X}) \in \mathbb{R}^{n_i \times \prod_{j\neq i} n_j}$.
- Using Corollary 3, we find $\beta_a = \left(\prod_i n_i\right)^{1/4} / \sqrt{\sum_i n_i}$.
- Coincides with $O\left(N^{\frac{d-2}{4}}\right)$ of **(Ben Arous et al, 2021)** for $n_i = N$.
- Same threshold for tensor power iteration initialized with tensor unfolding **(Auddy et al., 2021)**.