# Deciphering Asymmetric Spiked Tensor Models via Random Matrix Theory

## Abu Dhabi Stochastics Seminar

**Mohamed El Amine Seddik**
melaseddik.github.io

Joint work with M. Guillaud, R. Couillet, A. Decurninge
M. Mahfoud & M. Debbah

Technology Innovation Institute

**New York University Abu Dhabi - February 15th 2023**

# Outline

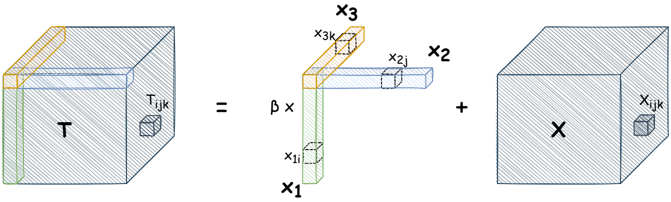
## Introduction
- Asymmetric Spiked Tensor Model
- Related Works
- Random Matrix Approach

## Analysis of the Asymmetric Spiked Tensor Model
- Tensors Singular Values and Vectors
- Associated Random Matrix
- Asymptotic Spectral Norm and Alignments

## Decomposition Algorithms and Complexity

## Generalization to Low-rank Spiked Tensor Models
- Hotteling-type Tensor Deflation
- Orthogonalized Tensor Deflation

# Introduction: Asymmetric Spiked Tensor Model




We consider the following model: $(\boldsymbol{x}_1 \otimes \boldsymbol{x}_2 \otimes \boldsymbol{x}_3)_{ijk} = x_{1i}x_{2j}x_{3k}$

$$\mathbf{T} = \underbrace{\beta \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_d}_{\text{signal}} + \frac{1}{\sqrt{n}} \underbrace{\mathbf{X}}_{\text{noise}} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$$

where $\beta \geq 0$, $\|\boldsymbol{x}_i\| = 1$, $X_{i_1 \ldots i_d} \sim \mathcal{N}(0,1)$ i.i.d. and $n = \sum_{i=1}^{d} n_i$.

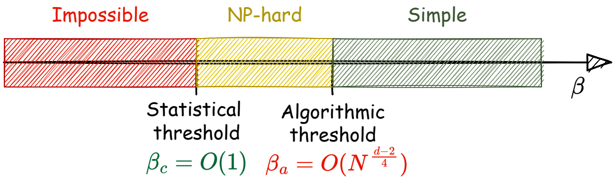
- Is it possible to recover the signal in theory? for which **critical** value of $\beta$?
- What **alignment** $\langle \boldsymbol{x}_i, \boldsymbol{u}_i \rangle$ between the signal and an estimator $\boldsymbol{u}_i(\mathbf{T})$?
- Is there an algorithm that can recover the signal in **polynomial time**?

# Related Works: Symmetric Case

Introduced initially by **(Montanari & Richard, 2014)**

$$\mathbf{Y} = \beta \boldsymbol{x}^{\otimes d} + \frac{1}{\sqrt{N}} \mathbf{W} \in \mathbb{R}^{N \times \cdots \times N}$$

where $\|\boldsymbol{x}\| = 1$ and **W** has random Gaussian entries and is **symmetric**.This is a natural extension of the classical spiked matrix model $\boldsymbol{Y} = \beta \boldsymbol{x}\boldsymbol{x}^\top + \frac{1}{\sqrt{N}} \boldsymbol{W}$.




Other works in the literature include: **(Montanari et al., 2015)**, **(Hopkins et al., 2020)**, **(Kim et al., 2017)**, **(Ben Arous et al., 2019)**, **(Jagannath et al, 2020)**, **(Perry et al., 2020)**, **(Ros et al., 2020)**, **(Goulart et al., 2022)**.

Of which **Goulart et al. "*A random matrix perspective on random tensors*", JMLR 2022.**

# Random Matrix Approach (Goulart et al., 2022)

The optimization problem of maximum likelihood estimator (MLE) for $d = 3$:

$$\min_{\lambda>0,\, \|\boldsymbol{u}\|=1} \left\| \mathbf{Y} - \lambda \boldsymbol{u}^{\otimes 3} \right\|_F^2 \quad \Leftrightarrow \quad \max_{\|\boldsymbol{u}\|=1} \langle \mathbf{Y}, \boldsymbol{u} \otimes \boldsymbol{u} \otimes \boldsymbol{u} \rangle$$

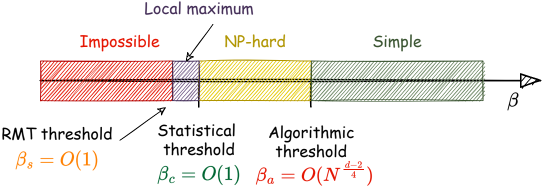
The critical points satisfy **(Lim, 2005)**:

$$\mathbf{Y}(\boldsymbol{u}, \boldsymbol{u}) = \lambda \boldsymbol{u} \quad \Leftrightarrow \quad \mathbf{Y}(\boldsymbol{u})\boldsymbol{u} = \lambda \boldsymbol{u}, \quad \|\boldsymbol{u}\| = 1$$

where $(\mathbf{Y}(\boldsymbol{u}, \boldsymbol{u}))_i = \sum_{jk} u_j u_k Y_{ijk}$ et $(\mathbf{Y}(\boldsymbol{u}))_{ij} = \sum_k u_k Y_{ijk}$. The MLE $\hat{\boldsymbol{x}}$ corresponds to the dominant eigenvector of $\mathbf{Y}(\hat{\boldsymbol{x}})$ : $\mathbf{Y}(\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} = \|\mathbf{Y}\|\hat{\boldsymbol{x}}$.

Hence, the approach from **(Goulart et al., 2021)** consists in studying:

$$\mathbf{Y}(\boldsymbol{u}) = \beta \langle \boldsymbol{x}, \boldsymbol{u} \rangle \boldsymbol{x}\boldsymbol{x}^\top + \frac{1}{\sqrt{N}} \mathbf{W}(\boldsymbol{u}) \in \mathbb{R}^{N \times N}$$



# Outline

# Tensors Singular Values and Vectors

The optimization problem of MLE for $d = 3$:

$$\min_{\lambda>0,\, \|\boldsymbol{u}_i\|=1} \|\mathbf{T} - \lambda \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3\|_F^2 \quad \Leftrightarrow \quad \max_{\prod_{i=1}^3 \|\boldsymbol{u}_i\|=1} \langle \mathbf{T}, \boldsymbol{u}_1 \otimes \boldsymbol{u}_2 \otimes \boldsymbol{u}_3 \rangle$$

The critical points satisfy **(Lim, 2005)**:

$$\mathbf{T}(\cdot, \boldsymbol{u}_2, \boldsymbol{u}_3) = \lambda \boldsymbol{u}_1,\ \mathbf{T}(\boldsymbol{u}_1, \cdot, \boldsymbol{u}_3) = \lambda \boldsymbol{u}_2,\ \mathbf{T}(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdot) = \lambda \boldsymbol{u}_3$$

where $\|\boldsymbol{u}_i\| = 1$ for all $i \in [3]$ and $(\mathbf{T}(\cdot, \boldsymbol{u}_2, \boldsymbol{u}_3))_i = \sum_{jk} u_{2j} u_{3k} T_{ijk}$.

- In contrast to the symmetric case, the choice of the associated *contraction* matrix is not straightforward. For instance:

$$\mathbf{T}(\boldsymbol{u}_3) \equiv \mathbf{T}(\cdot, \cdot, \boldsymbol{u}_3) = \beta \langle \boldsymbol{x}_3, \boldsymbol{u}_3 \rangle \boldsymbol{x}_1 \boldsymbol{x}_2^\top + \frac{1}{\sqrt{n}} \mathbf{X}(\cdot, \cdot, \boldsymbol{u}_3) \in \mathbb{R}^{n_1 \times n_2}$$

Objectives:

- Evaluate the asymptotic limits of $\hat{\lambda}$ and $\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i \rangle$ associated (a priori) to the MLE when $n_i \to \infty$.
- Define a *symmetric* random matrix that is equivalent to $\mathbf{T}$.

# Associated Random Matrix to T

**Stein's Lemma.** Let $X \sim \mathcal{N}(0,1)$, then $\mathbb{E}[Xf(X)] = \mathbb{E}[f'(X)]$.

Recall $\lambda = \mathbf{T}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \frac{1}{\sqrt{n}} \sum_{ijk} u_{1i} u_{2j} u_{3k} X_{ijk} + \beta \prod_{i=1}^{3} \langle \boldsymbol{x}_i, \boldsymbol{u}_i \rangle$.

$$\mathbb{E}[\lambda] = \frac{1}{\sqrt{n}} \sum_{ijk} \mathbb{E}\left[ u_{2j} u_{3k} \frac{\partial u_{1i}}{\partial X_{ijk}} \right] + \mathbb{E}\left[ u_{1i} u_{3k} \frac{\partial u_{2j}}{\partial X_{ijk}} \right] + \mathbb{E}\left[ u_{1i} u_{2j} \frac{\partial u_{3k}}{\partial X_{ijk}} \right] + \ldots$$

$$\begin{bmatrix} \frac{\partial \boldsymbol{u}_1}{\partial X_{ijk}} \\ \frac{\partial \boldsymbol{u}_2}{\partial X_{ijk}} \\ \frac{\partial \boldsymbol{u}_3}{\partial X_{ijk}} \end{bmatrix} \simeq -\frac{1}{\sqrt{n}} \Bigg( \underbrace{\begin{bmatrix} \mathbf{0}_{n_1 \times n_1} & \mathbf{T}(\boldsymbol{u}_3) & \mathbf{T}(\boldsymbol{u}_2) \\ \mathbf{T}(\boldsymbol{u}_3)^\intercal & \mathbf{0}_{n_2 \times n_2} & \mathbf{T}(\boldsymbol{u}_1) \\ \mathbf{T}(\boldsymbol{u}_2)^\intercal & \mathbf{T}(\boldsymbol{u}_1)^\intercal & \mathbf{0}_{n_3 \times n_3} \end{bmatrix}}_{\boldsymbol{\Phi}_3(\mathbf{T}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)} - \lambda \boldsymbol{I}_n \Bigg)^{-1} \begin{bmatrix} u_{2j} u_{3k} \boldsymbol{e}_i^{n_1} \\ u_{1i} u_{3k} \boldsymbol{e}_j^{n_2} \\ u_{1i} u_{2j} \boldsymbol{e}_k^{n_3} \end{bmatrix}$$

**The resolvent matrix:** $\boldsymbol{R}(z) = (\boldsymbol{\Phi}_3(\mathbf{T}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) - z\boldsymbol{I}_n)^{-1}$.
When $n_i \to \infty$, the non-vanishing terms involve the **trace** of $\boldsymbol{R}(z)$,

$$\lambda + \frac{1}{n} \operatorname{tr} \boldsymbol{R}(\lambda) = \beta \prod_{i=1}^{3} \langle \boldsymbol{x}_i, \boldsymbol{u}_i \rangle$$

# Associated Random Matrix to T

For an order-$d$ tensor the associated random matrix is $\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$

$$\Phi_d : (\mathbf{X}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_d) \longmapsto \begin{bmatrix} \mathbf{0}_{n_1 \times n_1} & \mathbf{X}^{12} & \mathbf{X}^{13} & \cdots & \mathbf{X}^{1d} \\ (\mathbf{X}^{12})^\top & \mathbf{0}_{n_2 \times n_2} & \mathbf{X}^{23} & \cdots & \mathbf{X}^{2d} \\ (\mathbf{X}^{13})^\top & (\mathbf{X}^{23})^\top & \mathbf{0}_{n_3 \times n_3} & \cdots & \mathbf{X}^{3d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathbf{X}^{1d})^\top & (\mathbf{X}^{2d})^\top & (\mathbf{X}^{3d})^\top & \cdots & \mathbf{0}_{n_d \times n_d} \end{bmatrix}$$

with $\mathbf{X}^{ij} \equiv \mathbf{X}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \cdot, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_{j-1}, \cdot, \boldsymbol{a}_{j+1}, \ldots, \boldsymbol{a}_d) \in \mathbb{R}^{n_i \times n_j}$.

Remark. $(d-1)\lambda$ is an eigenvalue of $\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$ with

$$\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d) \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_d \end{bmatrix} = (d-1)\lambda \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_d \end{bmatrix}$$

since $\mathbf{T}\left(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{j-1}, \cdot, \boldsymbol{u}_{j+1}, \ldots, \boldsymbol{u}_d\right) = \lambda \boldsymbol{u}_j$.

$$\operatorname{rank}(\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)) = \sum_{i=1}^d \min\left(n_i, \sum_{j \neq i} n_j\right)$$

# Spectral Measure of $\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$

Stieltjes Transform. The Stieltjes transform of a probability measure $\nu$ is $g_\nu(z) = \int \frac{d\nu(\lambda)}{\lambda - z}$, $z \in \mathbb{C} \setminus \mathcal{S}(\nu)$.

For $\boldsymbol{S} \in \mathrm{Sym}_n$ with $\lambda_i$ its eigenvalues and denote its resolvent $\boldsymbol{R}_{\boldsymbol{S}}(z) = (\boldsymbol{S} - z\boldsymbol{I}_n)^{-1}$, the ESM of $\boldsymbol{S}$ and its associated Stieltjes transform are:

$$\nu_{\boldsymbol{S}} = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i},\ g_{\nu_{\boldsymbol{S}}}(z) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda_i - z} = \frac{1}{n}\mathrm{tr}\,\boldsymbol{R}_{\boldsymbol{S}}(z),\ z \in \mathbb{C} \setminus \mathcal{S}(\nu_{\boldsymbol{S}})$$

Definition 1. Let $\nu$ by the probability measure with Stieltjes transform $g(z) = \sum_{i=1}^{d} g_i(z)$ verifying $\Im[g(z)] > 0$ for $\Im[z] > 0$, where $g_i(z)$ satisfies $g_i^2(z) - (g(z) + z)g_i(z) - c_i = 0$, for $z \notin \mathcal{S}(\nu)$.

Assumption 1. As $n_i \to \infty$ with $\frac{n_i}{\sum_j n_j} \to c_i \in (0, 1)$, there exists a sequence of critical points $(\hat{\lambda}, \hat{\boldsymbol{u}}_1, \ldots, \hat{\boldsymbol{u}}_d)$ s.t. $\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda$, $|\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i\rangle| \xrightarrow{\text{a.s.}} \rho_i$ with $\lambda \notin \mathcal{S}(\nu)$ and $\rho_i > 0$.

Theorem 1 (SGC'21). Under Assumption 1, the ESM of $\Phi_d(\mathbf{T}, \hat{\boldsymbol{u}}_1, \ldots, \hat{\boldsymbol{u}}_d)$ converges weakly to $\nu$ defined in Definition 1 (i.e. $\frac{1}{n}\mathrm{tr}\,\boldsymbol{R}(z) \xrightarrow{\text{a.s.}} g(z)$).

# Spectral Measure of $\Phi_d(\mathbf{T}, u_1, \ldots, u_d)$

Repeat

- $g_i \leftarrow c_i/(g_i - g - z)$
- $g \leftarrow \sum_i g_i$

Until convergence of $g$.

$$c_1 = \frac{c}{4}, \quad c_2 = \frac{c}{2}, \quad c_3 = 1 - \frac{3c}{4}$$

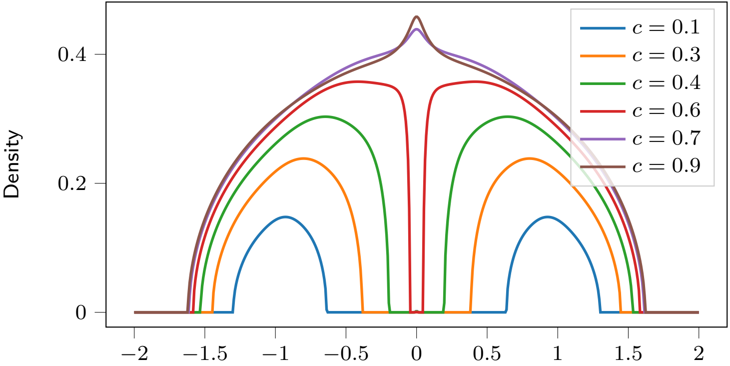



Figure: Density of the limiting spectral measure $\nu(dx) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im[g(x + i\epsilon)]$.

# Spectral Measure of $\Phi_d(\mathbf{T}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$

**Corollary 1.** When $c_i = \frac{1}{d}$ for all $i \in [d]$, the ESM of $\Phi_d(\mathbf{T}, \hat{\boldsymbol{u}}_1, \ldots, \hat{\boldsymbol{u}}_d)$ converges to a **semi-circle law** $\nu$ of support $\left[-2\sqrt{\frac{d-1}{d}}, 2\sqrt{\frac{d-1}{d}}\right]$, where

$$\nu(dx) = \frac{d}{2(d-1)\pi}\sqrt{\left(\frac{4(d-1)}{d} - x^2\right)^+}, \quad g(z) = \frac{-zd + d\sqrt{z^2 - \frac{4(d-1)}{d}}}{2(d-1)}$$

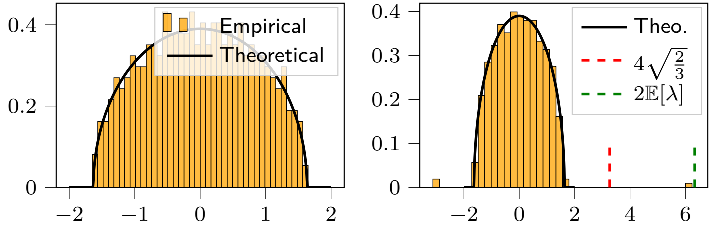



Figure: Spectrum of $\Phi_3(\mathbf{T}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ at initialization (left) and convergence (right) of tensor power iteration applied on $\mathbf{T}$. $n_1 = n_2 = n_3 = 150$ and $\beta = 3$.

$$\boldsymbol{u}_1 \leftarrow \frac{\mathbf{T}(\cdot, \boldsymbol{u}_2, \boldsymbol{u}_3)}{\|\mathbf{T}(\cdot, \boldsymbol{u}_2, \boldsymbol{u}_3)\|}, \quad \boldsymbol{u}_2 \leftarrow \frac{\mathbf{T}(\boldsymbol{u}_1, \cdot, \boldsymbol{u}_3)}{\|\mathbf{T}(\boldsymbol{u}_1, \cdot, \boldsymbol{u}_3)\|}, \quad \boldsymbol{u}_3 \leftarrow \frac{\mathbf{T}(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdot)}{\|\mathbf{T}(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdot)\|}$$

# Asymptotic Spectral Norm and Alignments

$$\mathbf{T}(\boldsymbol{x}_1, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{u}}_3) = \hat{\lambda}\langle \boldsymbol{x}_1, \hat{\boldsymbol{u}}_1\rangle \underbrace{\Rightarrow}_{\text{Stein}} \boxed{\left[\hat{\lambda} + g_2(\hat{\lambda}) + g_3(\hat{\lambda})\right] \langle \boldsymbol{x}_1, \hat{\boldsymbol{u}}_1\rangle = \beta \prod_{i=2}^{3} \langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i\rangle}$$

**Assumption 1.** As $n_i \to \infty$ with $\frac{n_i}{\sum_j n_j} \to c_i \in (0,1)$, there exists a sequence of critical points $(\hat{\lambda}, \hat{\boldsymbol{u}}_1, \dots, \hat{\boldsymbol{u}}_d)$ s.t. $\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda$, $|\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i\rangle| \xrightarrow{\text{a.s.}} \rho_i$ with $\lambda \notin \mathcal{S}(\nu)$ and $\rho_i > 0$.

**Theorem 2 (SGC'21).** For all $d \geq 3$, under Assumption 1, there exists $\beta_s > 0$ such that for all $\beta > \beta_s$

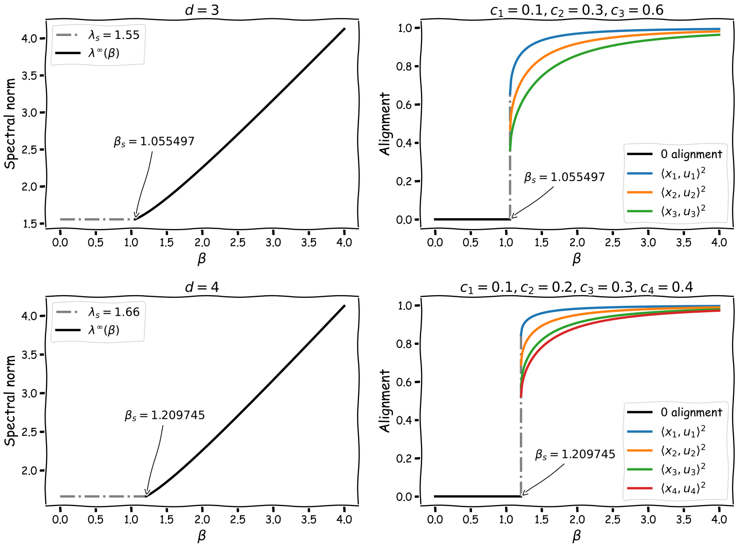
$$\hat{\lambda} \xrightarrow{\text{a.s.}} \lambda, \quad |\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i\rangle| \xrightarrow{\text{a.s.}} q_i(\lambda)$$

where $\lambda$ satisfies $f(\lambda, \beta) = 0$ with

$$f(z, \beta) = z + g(z) - \beta \prod_{i=1}^{d} q_i(z), \quad q_i(z) = \sqrt{1 - \frac{g_i^2(z)}{c_i}}$$

for $\beta \in [0, \beta_s]$, $\lambda$ is bounded and $|\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i\rangle| \xrightarrow{\text{a.s.}} 0$.

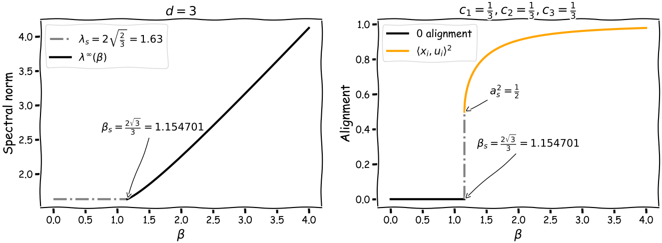
# Asymptotic Spectral Norm and Alignments



# Cubic Tensors

Corollary 2 (SGC'21). If $d = 3$ with $c_i = \frac{1}{3}$, then for all $\beta > \frac{2\sqrt{3}}{3}$

$$\begin{cases} \hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{\frac{\beta^2}{2} + 2 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{18\beta}} \\ |\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i \rangle| \xrightarrow{\text{a.s.}} \frac{\sqrt{9\beta^2 - 12 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{\beta}} + \sqrt{9\beta^2 + 36 + \frac{\sqrt{3}\sqrt{(3\beta^2-4)^3}}{\beta}}}{6\sqrt{2}\beta} \end{cases}$$




For hyper-cubic tensors of order $d$, we have

$$\beta_s = \sqrt{\frac{d-1}{d}\left(\frac{d-2}{d-1}\right)^{1-\frac{d}{2}}}, \quad \lim_{\beta \to \beta_s} \rho_i(\beta) = \sqrt{\frac{d-2}{d-1}}$$

# Spiked Matrix Model

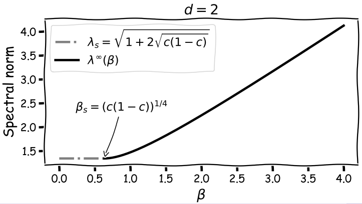
For $d = 3, n_3 = 1 \quad \Rightarrow \quad \boldsymbol{T} = \beta \boldsymbol{x}_1 \boldsymbol{x}_2^\top + \frac{1}{\sqrt{n_1 + n_2}} \boldsymbol{X} \in \mathbb{R}^{n_1 \times n_2}$

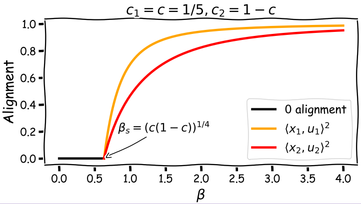
**Corollary 3 (SGC'21).** If $d = 3$ with $c_1 = c$ et $c_2 = 1 - c$ for $c \in [0, 1]$, the spiked tensor model becomes a **spiked matrix model** (i.e. $c_3 = 0$).

Let $\kappa(\beta, c) = \beta \sqrt{\frac{\beta^2(\beta^2+1) - c(c-1)}{(\beta^4 + c(c-1))(\beta^2 + 1 - c)}}$, for $\beta > \beta_s = \sqrt[4]{c(1-c)}$

$$\hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{\beta^2 + 1 + \frac{c(1-c)}{\beta^2}}, \quad |\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i \rangle| \xrightarrow{\text{a.s.}} \frac{1}{\kappa(\beta, c_i)}, \; i \in \{1, 2\}$$

while for $\beta \in [0, \beta_s]$, $\hat{\lambda} \xrightarrow{\text{a.s.}} \sqrt{1 + 2\sqrt{c(1-c)}}$ et $|\langle \boldsymbol{x}_i, \hat{\boldsymbol{u}}_i \rangle| \xrightarrow{\text{a.s.}} 0$.



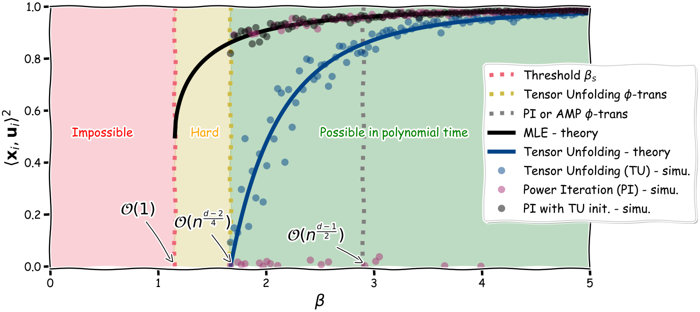



# Outline

# Decomposition Algorithms and Complexity

$$\min_{\lambda>0,\, \|\boldsymbol{u}_i\|=1} \|\mathbf{T} - \lambda \boldsymbol{u}_1 \otimes \cdots \otimes \boldsymbol{u}_d\|_F^2 \Rightarrow$$ NP-hard **(Hillar et al., 2013)**

- Tensor unfolding: $\mathcal{M}_i(\mathbf{T}) = \beta \boldsymbol{x}_i \boldsymbol{y}_i^\top + \frac{1}{\sqrt{n}} \mathcal{M}_i(\mathbf{X}) \in \mathbb{R}^{n_i \times \prod_{j \neq i} n_j}$.
- Using Corollary 3, we find $\beta_a = \left(\prod_i n_i\right)^{1/4} / \sqrt{\sum_i n_i}$.
- Coincides with $O\left(N^{\frac{d-2}{4}}\right)$ of **(Ben Arous et al, 2021)** for $n_i = N$.
- Same threshold for tensor power iteration initialized with tensor unfolding **(Auddy et al., 2021)**.



# Outline

# Hotteling-type Tensor Deflation

We consider the following rank-2 order-3 spiked tensor model

$$\mathbf{T}_1 = \sum_{i=1}^{2} \beta_i \boldsymbol{x}_{1i} \otimes \boldsymbol{x}_{2i} \otimes \boldsymbol{x}_{3i} + \frac{1}{\sqrt{n}} \mathbf{X} \in \mathbb{R}^{p \times p \times p}$$

where $\beta_i \geq 0$, $\|\boldsymbol{x}_{mi}\| = 1$, $X_{ijk} \sim \mathcal{N}(0,1)$ i.i.d. and $n = 3p$. Assume

$$\alpha \equiv \langle \boldsymbol{x}_{11}, \boldsymbol{x}_{12} \rangle = \langle \boldsymbol{x}_{21}, \boldsymbol{x}_{22} \rangle = \langle \boldsymbol{x}_{31}, \boldsymbol{x}_{32} \rangle \in [0,1]$$

Tensor Deflation. Compute $\hat{\lambda}_2 \hat{\boldsymbol{u}}_{12} \otimes \hat{\boldsymbol{u}}_{22} \otimes \hat{\boldsymbol{u}}_{32}$ as best rank-one approximation of $\mathbf{T}_2$ with

$$\mathbf{T}_2 = \mathbf{T}_1 - \hat{\lambda}_1 \hat{\boldsymbol{u}}_{11} \otimes \hat{\boldsymbol{u}}_{21} \otimes \hat{\boldsymbol{u}}_{31}$$

where $\hat{\lambda}_i \hat{\boldsymbol{u}}_{1i} \otimes \hat{\boldsymbol{u}}_{2i} \otimes \hat{\boldsymbol{u}}_{3i}$ is a critical point of

$$\arg\min_{\lambda_i > 0, \|\boldsymbol{u}_{mi}\| = 1} \|\mathbf{T}_i - \lambda_i \boldsymbol{u}_{1i} \otimes \boldsymbol{u}_{2i} \otimes \boldsymbol{u}_{3i}\|_F^2$$

Such a critical point satisfy

$$\mathbf{T}_i(\cdot, \hat{\boldsymbol{u}}_{2i}, \hat{\boldsymbol{u}}_{3i}) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{1i} \quad \mathbf{T}_i(\hat{\boldsymbol{u}}_{1i}, \cdot, \hat{\boldsymbol{u}}_{3i}) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{2i} \quad \mathbf{T}_i(\hat{\boldsymbol{u}}_{1i}, \hat{\boldsymbol{u}}_{2i}, \cdot) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{3i}$$

# Illustration of Signal Recovery with Deflation

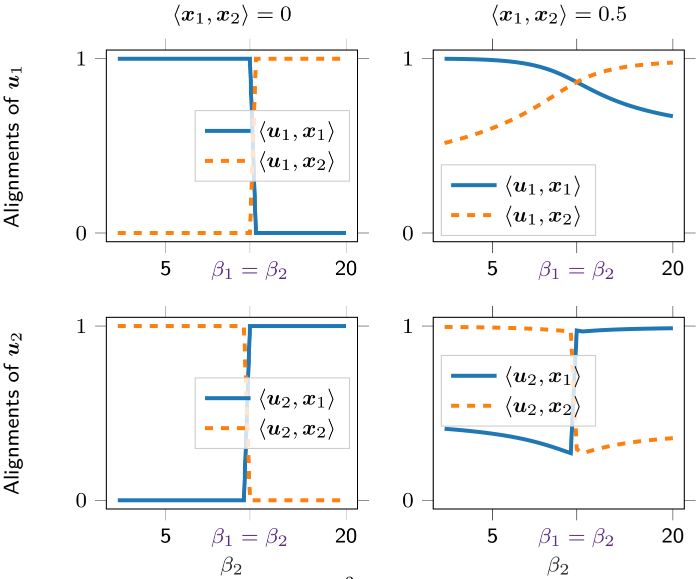



Figure: Deflation on $\mathbf{T}_1 = \sum_{i=1}^{2} \beta_i \boldsymbol{x}_i^{\otimes 3}$ with $\boldsymbol{x}_i = \boldsymbol{e}_i \in \mathbb{R}^p$.

# Associated Random Matrices

For both deflation steps:

$$\mathbf{T}_i \quad \to \quad \boldsymbol{\Phi}_3(\mathbf{T}_i, \hat{\boldsymbol{u}}_{1i}, \hat{\boldsymbol{u}}_{2i}, \hat{\boldsymbol{u}}_{3i}) \quad \to \quad \text{Stieltjes transform } g(z)$$

since $\mathbf{T}_i$'s are low-rank perturbations of $\frac{1}{\sqrt{n}}\mathbf{X}$.

Assumption 2. Assume that as $n \to \infty$, there exists a sequence of critical points $(\hat{\lambda}_i, \hat{\boldsymbol{u}}_{1i}, \hat{\boldsymbol{u}}_{2i}, \hat{\boldsymbol{u}}_{3i})$ such that

$$\hat{\lambda}_i \xrightarrow{\text{a.s.}} \lambda_i \quad |\langle \hat{\boldsymbol{u}}_{mi}, \boldsymbol{x}_{mj} \rangle| \xrightarrow{\text{a.s.}} \rho_{ij} \quad |\langle \hat{\boldsymbol{u}}_{m1}, \hat{\boldsymbol{u}}_{m2} \rangle| \xrightarrow{\text{a.s.}} \eta$$

with $\lambda_i > 2\sqrt{\frac{2}{3}}$ and $\rho_{ij}, \eta > 0$.

Theorem 3 (SGC'21). Under Assumption 2, the ESM of $\boldsymbol{\Phi}_3(\mathbf{T}_i, \hat{\boldsymbol{u}}_{1i}, \hat{\boldsymbol{u}}_{2i}, \hat{\boldsymbol{u}}_{3i})$ converges to the semi-circle law $\nu$ of compact support $\left[-2\sqrt{\frac{2}{3}}, 2\sqrt{\frac{2}{3}}\right]$, with Stieltjes transform

$$g(z) = \frac{-3z + 3\sqrt{z^2 - \frac{8}{3}}}{4}, \quad z > 2\sqrt{\frac{2}{3}}$$

# First Deflation Step

**Theorem 4 (SGD'22).** Under Assumption 2, $\lambda_1$, $\rho_{11}$ and $\rho_{12}$ satisfy

$$\begin{cases} f_g(\lambda_1) = \sum_{i=1}^2 \beta_i \rho_{1i}^3 \\ h_g(\lambda_1)\rho_{1j} = \sum_{i=1}^2 \beta_i \alpha_{ij} \rho_{1i}^2 \quad \text{for} \quad j \in [2] \end{cases}$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and $1$ otherwise, and denote $f_g(z) = z + g(z)$ and $h_g(z) = -\frac{1}{g(z)}$.

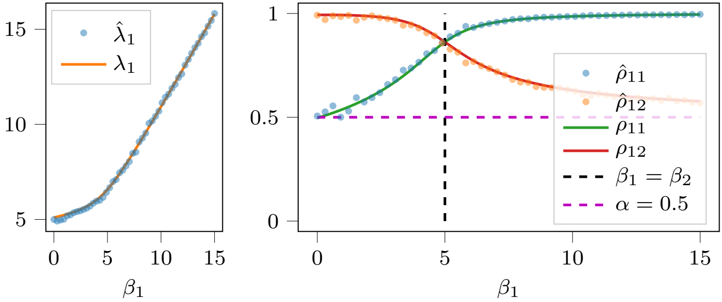



Figure: Simulated versus asymptotic singular value and alignments corresponding to the first deflation step. We considered $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$ and varying $\beta_1 \in [0, 15]$.

# Second Deflation Step

**Theorem 5 (SGD'22).** Under Assumption 2, $\lambda_1$, $\rho_{11}$ and $\rho_{12}$ satisfy

$$\begin{cases} f_g(\lambda_2) + \lambda_1 \eta^3 = \sum_{i=1}^2 \beta_i \rho_{2i}^3 \\ h_g(\lambda_2)\rho_{2j} + \lambda_1 \eta^2 \rho_{1j} = \sum_{i=1}^2 \beta_i \alpha_{ij} \rho_{2i}^2 \quad \text{for} \quad j \in [2] \\ h_g(\lambda_2)\eta + q_g(\lambda_1)\eta^2 = \sum_{i=1}^2 \beta_i \rho_{1i} \rho_{2i}^2 \end{cases}$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and 1 otherwise, and denote $f_g(z) = z + g(z)$, $h_g(z) = -\frac{1}{g(z)}$ and $q_g(z) = z + \frac{g(z)}{3}$.

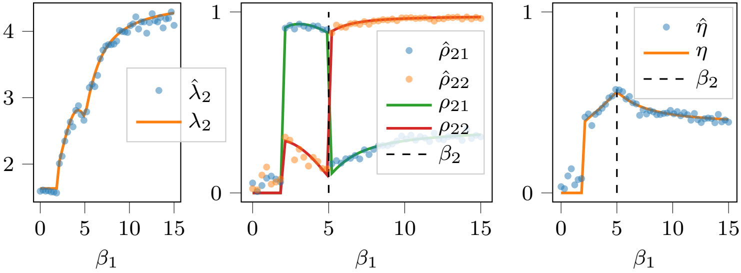



Figure: Simulated versus asymptotic singular value and alignments corresponding to the first deflation step. We considered $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$ and varying $\beta_1 \in [0, 15]$.

# Orthogonalized Tensor Deflation

We consider the following rank-2 order-3 spiked tensor model

$$\mathbf{T}_1 = \sum_{i=1}^{2} \beta_i \boldsymbol{x}_{1i} \otimes \boldsymbol{x}_{2i} \otimes \boldsymbol{x}_{3i} + \frac{1}{\sqrt{n}} \mathbf{X} \in \mathbb{R}^{p \times p \times p}$$

where $\beta_i \geq 0$, $\|\boldsymbol{x}_{mi}\| = 1$, $X_{ijk} \sim \mathcal{N}(0,1)$ i.i.d. and $n = 3p$. Assume

$$\alpha \equiv \langle \boldsymbol{x}_{11}, \boldsymbol{x}_{12} \rangle = \langle \boldsymbol{x}_{21}, \boldsymbol{x}_{22} \rangle = \langle \boldsymbol{x}_{31}, \boldsymbol{x}_{32} \rangle \in [0,1]$$

Orthogonalized Deflation. Compute $\hat{\lambda}_2 \hat{\boldsymbol{u}}_{12} \otimes \hat{\boldsymbol{u}}_{22} \otimes \hat{\boldsymbol{u}}_{32}$ as best rank-one approximation of $\mathbf{T}_2$ with

$$\mathbf{T}_2 \equiv \mathbf{T}_1 \times_1 \left( \boldsymbol{I}_p - \gamma \hat{\boldsymbol{u}}_{11} \hat{\boldsymbol{u}}_{11}^\top \right) = \mathbf{T}_1 - \gamma \hat{\boldsymbol{u}}_{11} \otimes \mathbf{T}_1(\hat{\boldsymbol{u}}_{11})$$

where $\gamma \in [0,1]$ and $\hat{\lambda}_i \hat{\boldsymbol{u}}_{1i} \otimes \hat{\boldsymbol{u}}_{2i} \otimes \hat{\boldsymbol{u}}_{3i}$ is a critical point of

$$\underset{\lambda_i > 0, \|\boldsymbol{u}_{mi}\| = 1}{\arg\min} \|\mathbf{T}_i - \lambda_i \boldsymbol{u}_{1i} \otimes \boldsymbol{u}_{2i} \otimes \boldsymbol{u}_{3i}\|_F^2$$

Such a critical point satisfy

$$\mathbf{T}_i(\cdot, \hat{\boldsymbol{u}}_{2i}, \hat{\boldsymbol{u}}_{3i}) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{1i} \quad \mathbf{T}_i(\hat{\boldsymbol{u}}_{1i}, \cdot, \hat{\boldsymbol{u}}_{3i}) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{2i} \quad \mathbf{T}_i(\hat{\boldsymbol{u}}_{1i}, \hat{\boldsymbol{u}}_{2i}, \cdot) = \hat{\lambda}_i \hat{\boldsymbol{u}}_{3i}$$

# Associated Random Matrix (Second Deflation Step)



Let $\hat\kappa = \langle \hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{12} \rangle$

$$\mathbf{T}_2 \to M_\gamma \equiv \frac{1}{\sqrt{n}} \begin{bmatrix} 0 & \mathbf{X}(\hat{\boldsymbol{u}}_{32}) & \mathbf{X}(\hat{\boldsymbol{u}}_{22}) \\ \mathbf{X}(\hat{\boldsymbol{u}}_{32})^\top & 0 & \mathbf{X}(\hat{\boldsymbol{u}}_{12}) - \gamma\hat\kappa \mathbf{X}(\hat{\boldsymbol{u}}_{11}) \\ \mathbf{X}(\hat{\boldsymbol{u}}_{22})^\top & \mathbf{X}(\hat{\boldsymbol{u}}_{12})^\top - \gamma\hat\kappa \mathbf{X}(\hat{\boldsymbol{u}}_{11})^\top & 0 \end{bmatrix}$$

**Remark.** If $\gamma = 1$ then $\hat\kappa = 0$.

$$\lambda_2 \langle \hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{12} \rangle = \mathbf{T}_2(\hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{22}, \hat{\boldsymbol{u}}_{32})$$
$$= \mathbf{T}_1(\hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{22}, \hat{\boldsymbol{u}}_{32}) - \underbrace{\langle \hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{11} \rangle}_{=1} \mathbf{T}_1(\hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{22}, \hat{\boldsymbol{u}}_{32}) = 0$$

which yields a **semi-circle law** as in Hotteling's deflation.

**Assumption 3.** Assume that as $n \to \infty$, there exists a sequence of critical points $(\hat\lambda_2, \hat{\boldsymbol{u}}_{12}, \hat{\boldsymbol{u}}_{22}, \hat{\boldsymbol{u}}_{32})$ such that for $m \neq 1$

$$\hat\lambda_2 \xrightarrow{\text{a.s.}} \lambda_2 \quad |\langle \hat{\boldsymbol{u}}_{12}, \boldsymbol{x}_{1i} \rangle| \xrightarrow{\text{a.s.}} \theta_{2i} \quad |\langle \hat{\boldsymbol{u}}_{m2}, \boldsymbol{x}_{mi} \rangle| \xrightarrow{\text{a.s.}} \rho_{2i}$$
$$|\langle \hat{\boldsymbol{u}}_{11}, \hat{\boldsymbol{u}}_{12} \rangle| \xrightarrow{\text{a.s.}} \kappa \quad |\langle \hat{\boldsymbol{u}}_{m1}, \hat{\boldsymbol{u}}_{m2} \rangle| \xrightarrow{\text{a.s.}} \eta$$

with $\lambda_2 > \lambda_+$ and $\theta_{2i}, \rho_{2i}, \eta > 0$.

# Associated Random Matrix (Second Deflation Step)

**Theorem 6 (SMD'23).** Under Assumption 3, the ESM of $\boldsymbol{M}_\gamma$ converges weakly to a deterministic measure $\mu$ having Stieltjes transform $s(z) = a(z) + 2b(z)$ verifying $\Im[s(z)] > 0$ for $\Im[z] > 0$, where $a(z)$ and $b(z)$ satisfy, for $z \notin \operatorname{Supp}(\mu)$

$$\begin{cases} [2b(z) + z]\, a(z) + \frac{1}{3} = 0 \\ (a(z) + z - \tau b(z)) b(z) + \frac{1}{3} = 0 \end{cases}$$

with $\tau = \gamma\kappa^2 - 1 + \kappa(\gamma - 1)$.

Repeat

- $a \leftarrow -1/(3(2b + z))$
- $b \leftarrow -1/(3(a + z - \tau b))$

Until convergence of $a$ and $b$.

$$\mu(dx) = \frac{1}{\pi} \lim_{\epsilon \to 0} \Im[s(x + i\epsilon)]$$

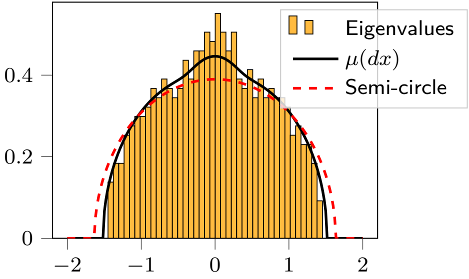



Figure: Histogram of the eigenvalues of $\boldsymbol{M}_\gamma$ and limiting measure $\mu$. We considered $p = 200$, $\beta_1 = 20$, $\beta_2 = 15$, $\alpha = 0.8$, $\gamma = 0.85$.

# Singular Value and Alignments (Second Deflation Step)



**Theorem 6 (SMD'23).** Under Assumption 3, $\lambda_2$, $\theta_{2i}$, $\rho_{2i}$, $\kappa$ and $\eta$ satisfy

$$\begin{cases} f_s(\lambda_2) - \frac{\gamma\kappa\eta^2}{3} g(\lambda_1) - 2\gamma\kappa^2 b(\lambda_2) = \sum_{i=1}^2 \beta_i \theta_{2i}\rho_{2i}^2 - \gamma\kappa \sum_{i=1}^2 \beta_i \rho_{1i}\rho_{2i}^2 \\ [f_s(\lambda_2) - a(\lambda_2)]\theta_{2j} - \gamma\rho_{1j}\left[\frac{\eta^2}{3} g(\lambda_1) + 2\kappa b(\lambda_2)\right] = \sum_{i=1}^2 \beta_i \alpha_{ij}\rho_{2i}^2 - \gamma\rho_{1j}\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 \\ [\lambda_2 + 2(1-\gamma)b(\lambda_2)]\,\kappa = (1-\gamma)\left[\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 - \frac{\eta^2}{3} g(\lambda_1)\right] \\ \left[f_s(\lambda_2) - (1+\gamma\kappa^2)b(\lambda_2)\right]\rho_{2j} = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}\alpha_{ij} - \gamma\kappa\left[\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}\alpha_{ij} - \frac{\rho_{1j}\eta}{3} g(\lambda_1)\right] \\ \left[\lambda_2 + a(\lambda_2) + (1-\gamma\kappa^2)b(\lambda_2) - \frac{\gamma\kappa}{3} g(\lambda_1)\right]\eta = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{1i}\rho_{2i} - \gamma\kappa\sum_{i=1}^2 \beta_i\rho_{1i}^2\rho_{2i} \end{cases}$$

where $\alpha_{ij} = \alpha$ if $i \neq j$ and $1$ otherwise, and denote $f_s(z) = z + s(z)$.

**Case $\gamma = 1$.** The above system reduces to

$$\begin{cases} f_g(\lambda_2) = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}^2 \\ h_g(\lambda_2)\theta_{2j} - \frac{\eta^2}{3} g(\lambda_1)\rho_{1j} = \sum_{i=1}^2 \beta_i\alpha_{ij}\rho_{2i}^2 - \rho_{1j}\sum_{i=1}^2 \beta_i\rho_{1i}\rho_{2i}^2 \\ h_g(\lambda_2)\rho_{2j} = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{2i}\alpha_{ij} \\ h_g(\lambda_2)\eta = \sum_{i=1}^2 \beta_i\theta_{2i}\rho_{1i}\rho_{2i} \end{cases}$$

since $\kappa = 0$.

# Singular Value and Alignments (Second Deflation Step)

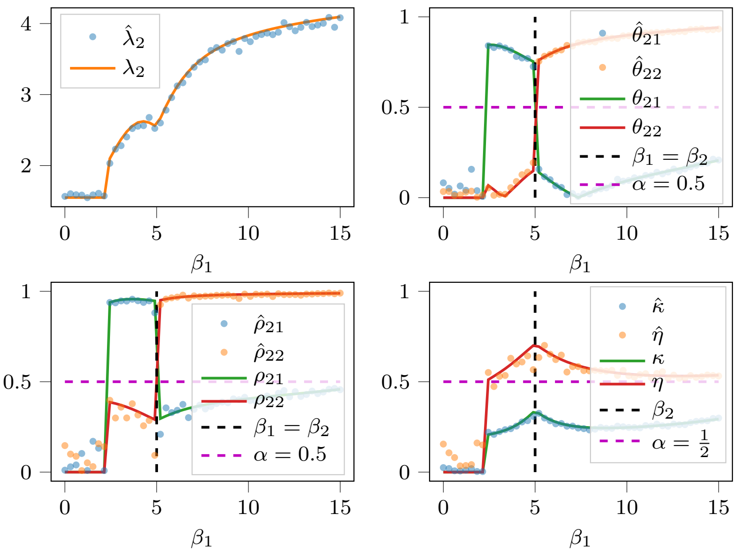



Figure: $\beta_2 = 5$, $\alpha = 0.5$, $p = 100$, $\gamma = 0.8$ and varying $\beta_1 \in [0, 15]$.

# Alignments Varying $\gamma$

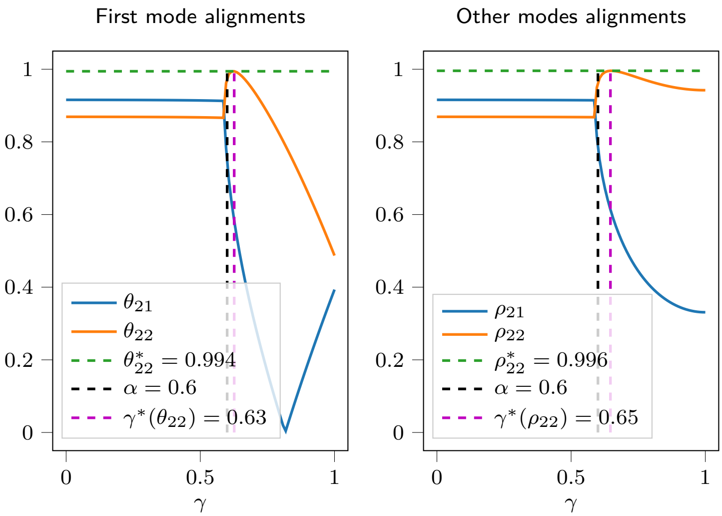



Figure: $\beta_1 = 10$, $\beta_2 = 9$, $\alpha = 0.6$, $p = 100$, $\alpha = 0.6$ and varying $\gamma \in [0, 1]$.

# Model Parameters Estimation

- Compute $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\eta} = \langle \hat{\boldsymbol{u}}_{21}, \hat{\boldsymbol{u}}_{22} \rangle$ by orthogonalized deflation for $\gamma = 1$.
- Solve the previous systems (for $\gamma = 1$) in $\beta_1$, $\beta_2$, $\alpha$ and the alignments.

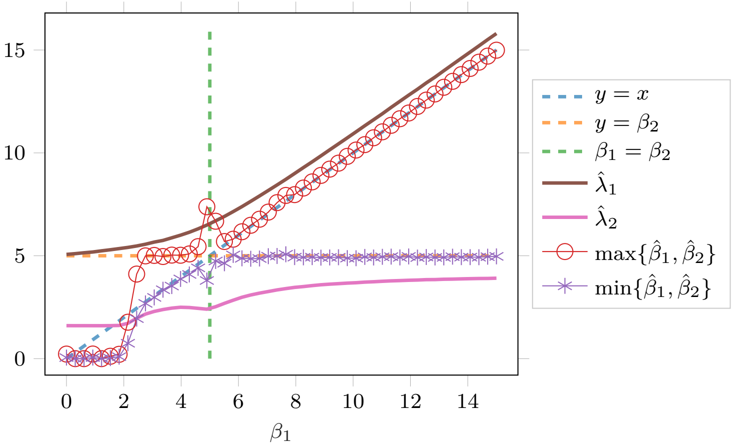



Figure: $\beta_2 = 5$, $\alpha = 0.5$, $p = 150$ and $\gamma = 1$ while varying $\beta_2$. The curves are averaged over $100$ realizations of $\mathbf{T}_1$.

# RTT-improved Deflation Algorithm

- Perform orthogonalized deflation with $\gamma = 1$.
- Model estimation $(\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$.
- Estimate optimal $\gamma^*$ to maximize $\rho_{22}$ (solve systems and update $\gamma \leftarrow \gamma - \epsilon$ for $\epsilon > 0$).
- Perform orthogonalized deflation with $\gamma^*$.
- Re-estimate first component as best rank-one approximation of $\mathbf{T}_2 - \min\{\hat{\beta}_1, \hat{\beta}_2\}\hat{u}_2^* \otimes \hat{v}_2^* \otimes \hat{w}_2^*$.

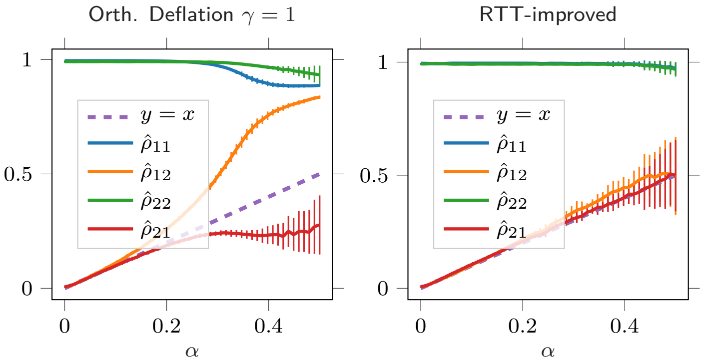
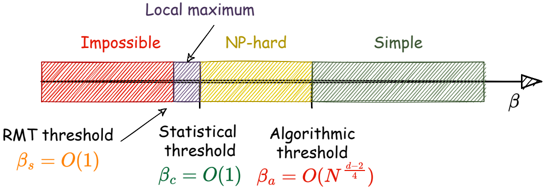



Figure: $\beta_1 = 6$, $\beta_2 = 5.7$ and $p = 150$. The curves are obtained by averaging over 200 realizations of $\mathbf{T}_1$.

# Take Away Messages

- The RMT approach allows the study of asymmetric spiked tensor models.
- The obtained results characterize the performance of the MLE for $\beta$ large enough (i.e., $\beta \geq \beta_c$).




## Open questions:

- Still unclear how to characterize the **phase transition** of the MLE with the RMT approach.
- Is it possible to find a **polynomial time algorithm** that is consistent below the computational threshold $\beta_a$?
- Study of higher order statistics and **fluctuations**?
- Proof of **consistency** of model estimation?
- Study the **existence** and **uniqueness** of the solutions of the deflation cases.
- **Universality** and generalization to other decomposition methods.

**Thank you for your attention!**

melaseddik.github.io

# References


Andrea Montanari and Emile Richard. "A statistical model for tensor PCA". In: *arXiv preprint arXiv:1411.1076* (2014)

José Henrique Goulart, Romain Couillet, and Pierre Comon. "A Random Matrix Perspective on Random Tensors". In: *stat* 1050 (2021), p. 2

Lek-Heng Lim. "Singular values and eigenvalues of tensors: a variational approach". In: *Proc. IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*. 2005, pp. 129–132

Christopher J Hillar and Lek-Heng Lim. "Most tensor problems are NP-hard". In: *Journal of the ACM (JACM)* 60.6 (2013), pp. 1–39

Gérard Ben Arous, Daniel Zhengyu Huang, and Jiaoyang Huang. "Long Random Matrices and Tensor Unfolding". In: *arXiv preprint arXiv:2110.10210* (2021)

Arnab Auddy and Ming Yuan. "On Estimating Rank-One Spiked Tensors in the Presence of Heavy Tailed Errors". In: *arXiv preprint arXiv:2107.09660* (2021)

Harold Hotelling. "Analysis of a complex of statistical variables into principal components". In: *Journal of Educational Psychology* (1933)

Mohamed El Amine Seddik, Maxime Guillaud, and Romain Couillet. "When Random Tensors meet Random Matrices". In: *arXiv preprint arXiv:2112.12348* (2021)

Mohamed El Amine Seddik, Maxime Guillaud, and Alexis Decurninge. "On the Accuracy of Hotelling-Type Tensor Deflation: A Random Tensor Analysis". In: *arXiv preprint arXiv:2211.09004* (2022)

Mohamed El Amine Seddik, Mohammed Mahfoud, and Merouane Debbah. "Optimizing Orthogonalized Tensor Deflation via Random Tensor Theory". In: *arXiv preprint arXiv:2302.05798* (2023)